On Vlasov approach to tokamaks near magnetic axis

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Abstract

A previous proof of non existence of tokamak equilibria with purely poloidal flow within macroscopic theory [Throumoulopoulos, Weitzner, Tasso, Physics of Plasmas **13**, 122501 (2006)] motivated this microscopic analysis near magnetic axis for toroidal and "straight" tokamak plasmas. Despite the new exact solutions of Vlasov's equation found here, the structure of macroscopic flows remains elusive.

1 Introduction

Some time ago (see [1, 2]), it was possible to prove non existence of tokamak equilibria with purely poloidal incompressible flow. Recently, an extension to compressible plasmas appeared in Ref.[3] including Hall term and pressure anisotropy. The proof for the incompressible case given in Refs.[1, 2] was global while the recent proof [3] is limited to the neighbouring of the magnetic axis through a kind of Mercier expansion.

This last result motivated the idea to extend the analysis to Vlasov-Maxwell equations examined near axis. An important ingredient is to write the Vlasov equation in cylindrical coordinates in a tokamak geometry, which simplifies the subsequent analysis. We use for that purpose the calculation done in an old ICTP report [4] where the Vlasov equation is written in arbitrary orthogonal coordinates.

In Section 2 the expression of the Vlasov equation is obtained in toroidal geometry. In Section 3 the ODEs of the characteristics are derived while Section 4 is devoted to "straight tokamaks" and Section 5 to discussion and conclusions.

2 Vlasov equation in orthogonal coordinates

As explained in Ref.[4] we consider a general system of orthogonal coordinates x^1, x^2, x^3 with the metric $ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$ and unit vectors $\mathbf{e}_i = \frac{\nabla x^i}{|\nabla x^i|}$ where i goes from 1 to 3. The velocity vector of a "microscopic" fluid element is then projected on the unit vectors $\mathbf{e}_i$ as

$$\mathbf{v} = v^i \mathbf{e}_i, \tag{1}$$

where the components v^i are independent upon space variables. The total derivative of $\mathbf{v}$ is

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{2}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields consistent with Maxwell equations and the charge to mass ratio $\frac{e}{m}$ is set to one. Projecting Eq.(2) on the unit vectors we obtain

$$\frac{dv^i}{dt} = \mathbf{e}_i \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{e}_i \cdot \mathbf{v} \times \nabla \times \mathbf{v}. \tag{3}$$

Finally, the Vlasov equation in orthogonal coordinates is given by

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{e}_i \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \frac{\partial f}{\partial v^i} + (\mathbf{e}_i \cdot \mathbf{v} \times \nabla \times \mathbf{v}) \frac{\partial f}{\partial v^i} = 0, \tag{4}$$

where f is a function of the x^i, v^i and time while $\mathbf{v}$ is given by Eq.(1). For more details see Ref. [4]. f stays here for the ion distribution while the distribution function for the electrons is governed by an equation similar to Eq.(4).

Let us now specialize on cylindrical coordinates $x^1 = r$, $x^2 = \phi$, $x^3 = z$. Then $\nabla \times \mathbf{e}_i = 0$ for $i = 1$ and 3 and $\nabla \times \mathbf{e}_2 = \mathbf{e}_1 \times \nabla\phi$. If we replace the indices 1, 2, 3 by r, ϕ, z we have $\nabla \times \mathbf{v} = v^\phi \mathbf{e}_r \times \nabla\phi$ and

$$\mathbf{v} \times \nabla \times \mathbf{v} = \frac{v^r v^\phi \mathbf{e}_\phi}{r} - \frac{(v^\phi)^2 \mathbf{e}_r}{r}. \tag{5}$$

So the last term of Eq.(4) becomes $[\frac{(v^\phi)^2}{r}\frac{\partial f}{\partial v^r} - \frac{v^r v^\phi}{r}\frac{\partial f}{\partial v^\phi}]$. Setting $\mathbf{B} = \mathbf{e}_\phi \frac{I}{r}$ near axis and $\frac{\partial f}{\partial t} = 0$ for steady state, Eq.(4) reads

$$\mathbf{v} \cdot \nabla f + (\mathbf{e}_i \cdot \nabla\Phi)\frac{\partial f}{\partial v^i} - \frac{[v^z I - (v^\phi)^2]}{r}\frac{\partial f}{\partial v^r} + \frac{v^r I}{r}\frac{\partial f}{\partial v^z} - \frac{v^r v^\phi}{r}\frac{\partial f}{\partial v^\phi} = 0. \tag{6}$$

Assuming $\nabla f = \nabla\Phi = 0$ on axis the final equation to solve is

$$-[v^z I - (v^\phi)^2]\frac{\partial f}{\partial v^r} - v^r v^\phi \frac{\partial f}{\partial v^\phi} + v^r I \frac{\partial f}{\partial v^z} = 0. \tag{7}$$

3 ODEs for characteristics

Let us start with the simpler case $I = 0$, then the characteristics of Eq.(7) are given by the solution of

$$-\frac{dv^r}{(v^\phi)^2} = \frac{dv^\phi}{v^r v^\phi}, \tag{8}$$

whose solution is $(v^r)^2 + (v^\phi)^2 = C$. Since $f = f(C, v^z) = f[((v^r)^2 + (v^\phi)^2), v^z]$ on axis we obtain for the toroidal flow

$$\int v^\phi f d^3\mathbf{v} = 0, \tag{9}$$

which means zero toroidal flow on axis.

For $I \neq 0$ the charasteristics are given by

$$-\frac{dv^r}{v^z I - (v^\phi)^2} = -\frac{dv^\phi}{v^r v^\phi} = \frac{dv^z}{v^r I}. \tag{10}$$

The last equality delivers $C_1 = v^z + I \ln |v^\phi|$, the second characteristic being the particle energy $C_2 = (v^r)^2 + (v^\phi)^2 + (v^z)^2$. C_1 is "antisymmetric" in v^z but symmetric in v^ϕ, which leads to

$$\int v^\phi f(C_1, C_2) d^3\mathbf{v} = 0, \int v^z f(C_1, C_2) d^3\mathbf{v} \neq 0. \tag{11}$$

It means that the ϕ-flow is zero while the unphysical z-flow is finite. This is obviously not acceptable.

4 "Straight" tokamaks

The straight tokamaks do have magnetohydrodynamic solutions with purely poloidal flow as known from previous work [5]. For the purpose of a microscopic theory the appropriate coordinate system is the cartesian one $x^1 = x$, $x^2 = y$, $x^3 = z$ so that the toroidal angular coordinate is replaced by y and the toroidal field I by B^y. Since $\nabla \times \mathbf{e}_i$ vanishes for all i, the term $\mathbf{v} \times \nabla \times \mathbf{v}$ in Eq.(4) disappears.

For the steady state with finite B^y, Eq.(7) is replaced by

$$-v^z \frac{\partial f}{\partial v^x} + v^x \frac{\partial f}{\partial v^z} = 0, \tag{12}$$

whose characteristic is given by

$$-\frac{dv^x}{v^z} = \frac{dv^z}{v^x}. \tag{13}$$

The solution of Eq.(13) is $C = (v^x)^2 + (v^z)^2$, which leads to $f = f((v^x)^2 + (v^z)^2, v^y)$. Purely poloidal flows are possible, which is consistent with Ref.[5].

5 Discussion and Conclusions

The result of section 3 obliges us to change the assumptions leading from Eq.(6) to Eq.(7) i.e. $\nabla f \neq 0$ instead of zero on the magnetic axis. The special canonical ϕ-momentum solution is of that kind, and leads naturally to toroidal flows but no poloidal flows. However, a comprehensive discussion of the problem cannot be done since the complete set of characteristics of Eq.(6) is not known.

Finally, though we know from section 3 that f must be a function of C_1 and C_2, we could, in addition, choose f to have different values for different signs of, for instance, v^ϕ. A known example of that kind of solutions is the case of BGK waves [6], in which the "free particles" have different distributions for different signs of their velocities. See also Ref.[7] for a quasi-neutral treatment. Though toroidal flows can then be constructed, physical constraints like isotropy of the pressure tensor or constraints on other moments or geometrical symmetries and, ultimately, collisions could exclude such solutions. Again we are led to look for the general solution of Eq.(6) with $\nabla f \neq 0$ on axis in order to discuss the structure of the macroscopic flows. Unfortunately, as mentioned before, the answer to this problem is quite uncertain.

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